

MATHEMATISCHES FORSCHUNGSMITTELE OBERWOLFACH

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**ON THE CONVERGENCE OF CLUSTER EXPANSIONS**SALVADOR MIRACLE-SOLE, CENTRE DE PHYSIQUE THÉORIQUE, MARSEILLE

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Let  $\mathcal{G}$  be a simple graph with a countable set  $\mathcal{P}$  of vertices, also called polymers. If  $\{\gamma, \gamma'\} \subset \mathcal{G}$  is an edge of the graph we say that the two polymers are incompatible and write  $\gamma \not\sim \gamma'$ . Otherwise they are compatible and write  $\gamma \sim \gamma'$ . A complex valued function  $\phi(\gamma)$ ,  $\gamma \in \mathcal{P}$ , is given. We call  $\phi(\gamma)$  the weight, or the activity, of the polymer  $\gamma$ . For any finite subset  $\Lambda \subset \mathcal{P}$ , the partition function  $Z(\Lambda)$  is defined by

$$Z(\Lambda) = \sum_{\substack{S \subset \Lambda \\ \text{compatible}}} \prod_{\gamma \in S} \phi(\gamma)$$

The sum runs over all subsets  $S$  of  $\Lambda$  such that  $\gamma \sim \gamma'$  for any two distinct elements of  $S$ . If  $S$  contains only one element it is considered as a compatible subset, and if  $S = \emptyset$ , the product is interpreted as the number 1.

A multi-index  $X$  on the set  $\mathcal{P}$  is a function  $X(\gamma)$ ,  $\gamma \in \mathcal{P}$ , taking non-negative integer values and such that  $\text{supp } X$  is a finite non empty set. Then

$$\ln Z(\Lambda) = \sum_{X, \text{supp } X \subset \Lambda} a^T(X) \prod_{\gamma \in \mathcal{P}} \phi(\gamma)^{X(\gamma)} = \sum_{X, \text{supp } X \subset \Lambda} \phi^T(X)$$

where the coefficients  $a^T(X)$  depend only on  $X$  (not on  $\Lambda$ ), and  $a^T(X) \neq 0$  only if  $X$  is a cluster, i.e., only if the restriction of  $\mathcal{G}$  to the vertices in  $\text{supp } X$  is a connected graph.

**THEOREM:** Assume that there is a positive function  $\mu(\gamma)$ ,  $\gamma \in \mathcal{P}$ , such that, for all  $\gamma_0 \in \mathcal{P}$ ,

$$|\phi(\gamma_0)| \leq (e^{\mu(\gamma_0)} - 1) \exp \left( - \sum_{\gamma \not\sim \gamma_0} \mu(\gamma) \right)$$

Then, for all  $\gamma_1 \in \mathcal{P}$ , we have

$$\begin{aligned} \sum_{X, X(\gamma_1) \geq 1} |\phi^T(X)| &\leq \mu(\gamma_1) \\ \sum_X X(\gamma_1) |\phi^T(X)| &\leq e^{\mu(\gamma_1)} - 1 \end{aligned}$$

The theorem is proved by an induction argument using the following relation: If  $\gamma_0 \notin \Lambda$ , then

$$Z(\Lambda \cup \{\gamma_0\}) = Z(\Lambda) + \phi(\gamma_0) Z(\Lambda_0)$$

with  $\Lambda_0 = \{\gamma \in \Lambda : \gamma \sim \gamma_0\}$ .

More details and references can be found in S. Miracle-Sole, *Physica A* **279**, 244–249, 2000.

**NOTE:** To any multi-index  $X$  we associate a graph  $g(X)$  with  $|X| = \sum_{\gamma} X(\gamma)$  vertices:  $X(\gamma_i)$  distinct vertices are associated to  $\gamma_i$ , for each  $\gamma_i \in \text{supp } X$ . We draw an edge between the vertices associated to  $\gamma_i$  and  $\gamma_j$  if  $\gamma_i \not\sim \gamma_j$ , and also if  $\gamma_i = \gamma_j$ . We denote by  $\mathcal{G}(X)$  the set of all connected subgraphs of  $g(X)$  whose vertices coincide with those of  $g(X)$ . The number of edges of a graph  $g$  is denoted by  $|g|$ . If  $|X| = 1$  we interpret  $\mathcal{G}(X)$  as having one subgraph with  $|g| = 0$ . Then  $a^T(X) = 0$  if  $g(X)$  is not connected, otherwise:

$$a^T(X) = \left( \prod_{\gamma \in \mathcal{P}} X(\gamma)! \right)^{-1} \sum_{g \in \mathcal{G}(X)} (-1)^{|g|}$$

This formula, and the expansion of  $\ln Z(\Lambda)$ , was proved by G. Gallavotti, A. Martin-Lof and S. Miracle-Sole in: “Mathematical Methods in Statistical Mechanics”, A. Lenard, ed., pp. 162–202, Springer, Berlin, 1973.